Flowing windowpanes: fact or fiction?

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Glass has properties of a liquid. But do glass windows really flow over centuries, becoming thicker at the bottom, as is commonly reported? Simple calculations show that the time $t$ taken for a windowpane of height $L_0$ to increase in thickness by $q\%$ due to gravity $g$ is given by

$$t = \frac{4\mu}{\rho g L_0} \frac{q}{100},$$

where the glass has viscosity $\mu$ and density $\rho$. For the small windowpanes common in medieval times this amounts to some millions of years! Thus, window glass behaves as a solid.

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1. Introduction

It is commonly reported, both in scientific circles and throughout the wider community, that apparently solid glass at normal ambient-air temperatures is in fact a liquid. The evidence usually given for this is that centuries-old windowpanes have greater thickness at the bottom than the top due to very slow downwards flow of the glass over the centuries (Budinski 1996). Certainly, at a molecular level, glass has the properties of, and can be considered, a liquid (Morey 1938). However, this ‘proof’ is a myth and, for practical purposes, glass behaves as a solid that will not flow under the influence of gravity.

The methods of manufacturing window glass before the 1959 float-glass process provide ready explanations for observed thickness variations in windowpanes and it is likely that panes were mounted, for structural reasons, with any thicker edge at the bottom (Plumb 1989). Zanotta (1998) refutes the myth with reference to relaxation times (in excess of $10^{32}$ years) for typical medieval glasses. But still the myth is widely regarded as credible so further discussion is evidently warranted. Here, we disprove this popular belief in a simple manner that, perhaps, will be easily understood.

We suppose that window glass is an extremely viscous liquid that will flow. This flow will be described by standard fluid-flow equations. An approximate solution to these can be easily obtained using a simple and well-known mathematical method, from which we may compute the time-scale required for any appreciable thickening at the base of the windowpane.

2. A mathematical model

A typical windowpane is thin relative to both its breadth and height. This enables us to consider just a vertical section through the glass, as shown in figure 1. The height...
of the windowpane is given, as a function of time, by \( x = L(t) \), and its two faces are given by \( y = \pm h(x, t) \). Initially, the two faces are vertical planes \( y = \pm h(x, 0) = \pm h_0 \) and the window height is \( L_0 \). Since a windowframe does not prevent small movement along the \( y \)-axis (as witnessed by the fact that a window will rattle in a reasonably high wind), the windowframe is assumed to provide a support at \( x = 0 \) that prevents downwards flow of the glass but allows horizontal flow. All other boundaries of the glass are considered to be completely free surfaces.

Considering the glass to be a very viscous liquid, its flow will satisfy (Stokes 1998) the creeping-flow equations

\[
\begin{align*}
px &= \mu(u_{xx} + u_{yy}) - \rho g, \\
py &= \mu(v_{xx} + v_{yy}),
\end{align*}
\]

and the mass-conservation (i.e. continuity) equation

\[
ux + vy = 0.
\]

Here, \( p, u \) and \( v \) are the pressure and \( x, y \)-components of velocity respectively, subscripts indicate differentiation with respect to the subscript variable, \( \rho \) is the glass density, \( \mu \) its viscosity and \( g \) is gravitational acceleration. A free surface having normal \( n = (nx, ny) \) cannot be subject to surface stresses, a condition which is indicated by

\[
\begin{align*}
-pnx + \mu[2nxu_x + ny(u_y + v_x)] &= 0, \\
-pny + \mu[nx(u_y + v_x) + 2nyv_y] &= 0.
\end{align*}
\]

Furthermore, on a free surface the so-called kinematic condition, which states that all fluid particles initially on the boundary remain there, must also be satisfied. At the top edge of the window this simply requires that \( u = L_t \), while on the window faces it leads to

\[
v = \pm h_t \pm uh_x.
\]
The final condition that must be satisfied is

\[ u = 0 \quad \text{at} \quad x = 0. \quad (2.7) \]

Now, since the window is thin and symmetrical about \( y = 0 \) for all time, we may write \( u, v \) and \( p \) as the following Taylor expansions:

\[
\begin{align*}
U(x, y, t) &= u_0(x, t) + y^2 u_2(x, t) + \ldots, \\
v(x, y, t) &= y v_1(x, t) + y^3 v_3(x, t) + \ldots, \\
p(x, y, t) &= p_0(x, t) + y^2 p_2(x, t) + \ldots.
\end{align*}
\]

Then, using standard perturbation methods (Van Dyke 1975), we substitute these expressions into (2.1)–(2.7) and equate like powers of \( y \). Thus, mass conservation (2.3) immediately requires

\[ v_1 = -u_0x; \quad (2.8) \]

on the free surfaces \( y = \pm h \) having normals \( \mathbf{n} = (\mp h_x, 1) \), condition (2.5) together with result (2.8) shows that

\[ p_0 = -2\mu u_0x; \quad (2.9) \]

and from the creeping-flow equation (2.1) and result (2.9) we have

\[ 2u_2 = \frac{\rho g}{\mu} - 3u_0xx. \]

Next, satisfying condition (2.4) on \( y = \pm h \), and the kinematic conditions (2.6), and substituting the expressions derived for \( p_0, v_1 \) and \( u_2 \), finally yields

\[
\begin{align*}
h \frac{\rho g}{\mu} &= 4(h u_0x)_x, \quad (2.10) \\
h_t &= -(u_0h)_x. \quad (2.11)
\end{align*}
\]

Thus, the flow and changing shape of the window glass is, to leading order, obtained by solving the coupled equations (2.10) and (2.11) for \( u_0(x, t) \) and \( h(x, t) \) subject to suitable boundary conditions. One of these is (2.7), and we need one further condition at \( x = L(t) \) which is obtained from the zero-stress condition (2.4) with \( \mathbf{n} = (1, 0) \) (i.e. \( -p + 2\mu u_x = 0 \)) to give

\[ u_{0x} = 0. \quad (2.12) \]

A similar method has been used to model the drawing of thin viscous fibres and sheets (as in the manufacture of optical glass fibres and sheet glass) leading to equations similar to (2.10) and (2.11) but without any gravitational term (Dewynne et al. 1989; Howell 1996). Equations identical to (2.10) and (2.11), but with the direction of gravity reversed, are obtained in the context of gravity-driven extensional flows such as dripping honey (Stokes et al. 1999).

3. The solution

The solution to the problem defined above is, in one sense, most easily obtained by converting to Lagrangian coordinates as described in Stokes et al. (1999). We do
not repeat the details here, but just write down the solution, which may be directly obtained from the equations given in Stokes et al. (1999). It is

\[
h(x, t) = h_0 \left[ 1 + \frac{\rho gtL_0}{4\mu} \right] \exp \left( -\frac{\rho gt x}{4\mu} \right)
\]

for \( 0 \leq x \leq L(t) \), \( L(t) = \frac{4\mu}{\rho gt} \log \left( 1 + \frac{\rho gtL_0}{4\mu} \right) \).

However, for an initially rectangular windowpane, the following method, which requires only basic calculus, is simple and gives an approximate solution that yields all the necessary information for our current purposes.

First, we note that the time period over which glass flows is large, so that the life span of even centuries-old window glass is relatively short and variations in the glass thickness will be small. Thus, we are justified in considering the solution at time \( t = 0^+ \) when, for practical purposes, \( h \approx h_0 \), \( h_x \approx 0 \) and \( L \approx L_0 \). Then, (2.10) gives

\[
u_{0xx} = \frac{\rho g}{4\mu},
\]

which, on integrating and satisfying \( u_{0x} = 0 \) at \( x = L_0 \), in turn gives

\[u_{0x} = \frac{\rho gL_0}{4\mu} \left( \frac{x}{L_0} - 1 \right).\]

From (2.11) we have

\[h_t = -h_0 u_{0x} = h_0 \frac{\rho gL_0}{4\mu} \left( 1 - \frac{x}{L_0} \right),\]

and on integrating with respect to time \( t \) we obtain

\[h(x, t) = h_0 \left[ 1 + \frac{\rho gL_0}{4\mu} \left( 1 - \frac{x}{L_0} \right) t \right].\]  

This last equation may be obtained from the small-time Taylor expansion of (3.1). It correctly tells us that the glass thickness at the top of the window does not change over time, i.e. \( h = h_0 \) at \( x = L_0 \). It also correctly tells us that, for all (not necessarily small) time \( t > 0 \), the maximum windowpane thickness is at its base \( x = 0 \) and is given by

\[2h(0, t) = 2h_0 \left( 1 + \frac{\rho gL_0}{4\mu} t \right).\]

Note that (3.3) is obtained from (3.1) on setting \( x = 0 \). Thus, an increase in thickness of \( q\% \) occurs in a time of

\[t = \frac{4\mu}{\rho gL_0} \frac{q}{100}.
\]

Any increase in windowpane thickness at its base must, of course, be accompanied by a decrease in the windowpane height. The change in height after a time \( t \) can be determined from the expression for \( L(t) \) given in (3.1). Alternatively, and in keeping with our simple approach so far, we may find an approximate value using the fact that mass conservation demands that the cross-sectional area of the window remains
constant. Now, (3.2) indicates that the cross-section is (approximately) trapezoidal
in shape, from which we have

\[ L_0 h_0 = L(t) \frac{h_0}{2} \left( 2 + \frac{\rho g L_0 \mu}{4} t \right). \]

So, an increase in windowpane thickness of \( q\% \) at the base, after a time \( t \) given by
(3.4), is accompanied by a decrease in the windowpane height of about

\[ L_0 - L(t) = L_0 \left( \frac{q/100}{2 + q/100} \right). \] (3.5)

Now, (soda-lime) window glass is substantially rigid at a temperature of about
500 °C when the viscosity is of order \( 10^{14.5} \) poise (Holloway 1973; Scholze & Kreidl
1986). As the temperature decreases, the viscosity increases enormously and at
normal room temperatures we can expect the viscosity to be around \( 10^{20} \) poise
\( (10^{19} \text{ Pa s}) \) or even greater. With the density of glass at about 2500 kg m\(^{-3}\), the
time taken for a windowpane with an initial height of (say) 250 mm to increase just
5% in thickness is at least (see equation (3.4))

\[ 4 \times 0.05 \frac{\mu}{\rho g L_0} = 0.2 \times \frac{10^{19}}{2500 \times 10 \times 0.25} \]
\[ = 3.2 \times 10^{14} \text{ s} \]
\[ \approx 10^7 \text{ years.} \]

Thus, we must wait some 10 million years (!) and not centuries for a windowpane
with initial height and thickness of 250 mm \( \times \) 5 mm to thicken by just 0.25 mm at
the bottom. At this time, (3.5) also tells us that the height of the windowpane will
be over 6 mm smaller, leaving a significant gap between the top of the pane and
the windowframe, if indeed the glass has not fallen out of the frame. This sizable
decrease in height is a necessary consequence of glass flow in windowpanes which,
from all reports, has not been observed. Modern windows with heights of a few
metres will still require around one million years for a 5% increase in thickness, and
the corresponding decrease in height will be numbers of centimetres.

In conclusion, it is clear that the story of viscous-flow of old glass windowpanes
resulting in greater thickness at the bottom than the top is false.

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