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An optimal speed for traversing a constant rain

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Whether it is better to walk or run through a rain shower is a question often posed by adult and child alike. The answer to this question requires careful physical analysis and provides an instructive, even illuminating exercise for high school as well as undergraduate introductory physics classes. In this paper, we will derive a rule for the optimal travel speed which minimizes the number of rainstrikes on a given surface element over a given course. We shall also investigate the speed versus intensity relationship. Methods of extending these results to include rainfalls with horizontal velocity relative to the travel path and for surfaces of arbitrary geometry are presented in Sec. VI.

I. INTRODUCTION

Is it preferable to walk or run through the rain, indeed, does one's speed make a difference? At first brush, intuition tells us to run as fast as possible in order to spend a minimum of time in the rain. Experience (e.g., having a run through a downpour), however, suggests that slower speeds may be superior, since the "apparent intensity" of the rain rapidly increases with speed. Clearly, however, zero velocity would eventually lead to saturation, regardless of the intensity of the rain. Somewhere between these extremes there should lie a speed which will result in a minimum number of rainstrikes on a body of arbitrary geometry. In this paper, we will derive a strategy for minimizing the number of rainstrikes encountered while traveling through a constant stationary rainfall. Several consequences of this optimal solution will be discussed, and the results then generalized for moving rainfalls and bodies of complex geometry.

II. RELATION OF TRAVEL VELOCITY TO THE NUMBER OF RAINSTRIKES

Consider a horizontally isotropic and time-invariant rainstorm of uniform intensity $N$ drops per second per unit area. We need not assume all the raindrops are of equal mass, but rather that the average mass crossing the unit area per second is constant. We will further assume that all raindrops are falling at equal terminal velocity; this assumption is not as restrictive as it might seem, since drop volume and hence aerodynamic cross section remain unconstrained. We now choose the vertical axis to be positive downward, and assume the rain falls parallel to the gravity vector and normal to a horizontal surface on which all motion takes place (Fig. 1). Later, these constraints will be relaxed.

For a square planar surface element of area $A$ oriented at some angle $\theta$ to local vertical (and hence the rain), the number of rainstrikes impacting the element $A$ in a given time $t$ is

$$S = S^r + S^f,$$

where $S = \text{the total number of rainstrikes}$, $S^r = \text{the total number of rainstrikes per unit time (the strike rate)}$, $S^f = \text{the number of strikes impacting the horizontal (i.e., parallel to the surface) cross section of } A \text{ per unit time}$, $S^z = \text{the number of strikes impacting the vertical (i.e., parallel to the local vertical, } Z \text{) cross section of } A \text{ per unit time}$. Since the rain falls normal to the plane of travel, $S^f$ is given by

$$S^f = N A^* = N A \sin \theta.$$

Now $S^z$ is governed by the vertical cross section $A_z = A \cos \theta$, as well as the horizontal rate of travel through the rain, $v_H$. We will assume the convention $v_H > 0$ in the $+X$ direction; since the rain is isotropic, the initial choice of a $+X$ direction is arbitrary. Note that $S^z$ is the rate at which raindrops are "swept up" by virtue of the surface element's motion through the rain field. The vertical height of the element $A$ is just $Z_A = \sqrt{A} \cos \theta$. We now calculate the volume $V_A$ swept out by the area $\sqrt{A} \cos \theta$ by taking the product of $Z_A$ and $Av_H t$

$$V_A = \sqrt{A} Z_A v_H t = (A v_H \cos \theta)t.$$  

One should carefully note that upper case $V$'s denote volumes, while lower case $v$'s denote velocities throughout this paper. Obviously, raindrops falling for $t$ seconds at constant terminal velocity $v_z$ will fall a distance $Z_R = v_z t$. The volume thus occupied by $N$ raindrops of intensity $N$ drops/cm$^2$/s during the same $t$ seconds is

$$V_R = Z_R/N = v_z t / N,$$

where the downward positive orientation of the vertical axis insures that the rain density will be positive for falling rain in this model.
Dividing the volume element $V_A$ by $V_R$ will remove the effect induced by horizontal motion across the rain field from the mathematics. The total number of drops swept up by the vertical cross section of the surface element in the given time (Fig. 2) is thus given by,

$$S'_2 = NA\left(\frac{v_h}{v_z}\right) \cos \theta \quad \text{(drops/unit time)} \quad (4)$$

and the total rate

$$S' = NA\left[\sin \theta + \left(\frac{v_h}{v_z}\right) \cos \theta \right] . \quad (5)$$

so that

$$S = NA\left[\sin \theta + \left(\frac{v_h}{v_z}\right) \cos \theta \right] t . \quad (6)$$

The time to travel a horizontal traverse of distance $D_H$ at constant velocity $v_H$ is just $t = D_H/v_H$, so the total number of rainstrikes encountered over the path length $D_H$ is

$$S(v_h, v_z, \theta, D_H) = NA\left[\sin \theta + \frac{v_H}{v_z} \cos \theta \right] D_H . \quad (7)$$

Equations (5)–(7) describe the physics; let us review their significance so that we may minimize the rainstrike count. Reflection on the term due to $S_H$ demonstrates that maximizing the velocity through the rain will be rainstrike minimizing (i.e., the less time spent in the rain, the better). Further, the $S'_2$ term is independent of the horizontal traverse speed in our idealized rain. Although $S'$, the perceived intensity of the rainstrikes per unit time, will increase with velocity $v_H$ [cf. Eq. (5)], the total number of raindrops a frontal cross section $A_Z$ must intersect over $D_H$ will remain invariant. Notice that, in order to minimize the total number of strikes on a given body, it is necessary to minimize the vertical cross section. This holds, because only drops striking the frontal area can be influenced by the forward velocity. Although this may seem contradictory to common sense, it confirms the advisability of leaning into the rain (increasing $A_H$ at the expense of $A_Z$) and proceeding across $D_H$ with all possible haste. We conclude that traveling faster does indeed assure one will arrive drier!

From Eq. (7) we have

$$S = N\left(A_H/v_H + A_Z/v_Z\right) D_H . \quad (7a)$$

The original question, “Is it better to walk or run through the rain?”, can now be answered. Notice in the limiting case as $v_H \to \infty$, $S \to ND_H \left(A_Z/v_Z\right)$, thus eliminating strikes on the horizontal projection of $A$. Lying prone will minimize the remaining vertical cross section. Taken together, these two steps minimize $S$, the number of rainstrikes encountered over the traverse. Taken to its logical extreme, one should elect to ride a skateboard horizontally through the rain as rapidly as possible!

III. RAINSTRIKE MINIMIZING SOLUTION—TWO INTERESTING CONSEQUENCES

Does the size of the raindrops affect the solution obtained in Eq. (7)? Let us now adopt the terminal velocity of the rain for $v_z$, which will contribute inversely to the number of strikes received on the vertical cross section. Rainfall terminal velocity is governed by a complicated but monotonically increasing function of equivalent spherical radius, $R$. Larger drops fall faster, with $v_z$ values ranging from 5 to 8 m/s; those drops with $R$ greater than about 3 to 4 mm will be unstable against the turbulent shear forces created by their fall and will break into smaller droplets with correspondingly smaller terminal velocities. Therefore, larger drops falling at higher velocities will cover more vertical distance per unit time and hence will create a larger drop volume element than smaller drops; this “stretching” of the volume element will reduce the number of drops per unit vertical distance. Thus for equal intensity rainstorms, one receives less rainstrikes in a storm with larger drops. But does one stay drier? For drops with radius greater than about 19 μm, raindrop terminal velocities increase at a slower pace than does drop volume. Hence the total water mass striking a surface element increases as a function of drop radius. Summarizing these results: larger raindrops will reduce the number of strikes received, but increase the overall water volume encountered.

Yet another interesting consequence of the solution presented in Eq. (1) concerns rainstrike intensity. Frequently one is not concerned with the total number of strikes received but with the number of strikes per unit time. This is often the case for automobile drivers as well as ship and aircraft pilots who are sheltered from the water but are concerned with visibility. As evidenced by Eq. (5), if one is constrained to horizontal motion, only the intensity due to forward motion can be controlled. Slower forward speeds will reduce this intensity to acceptable levels. To maintain constant intensity, one should vary $v_H$ inversely with rainfall intensity; this strategy is commonly practiced by automobile drivers. Although aircraft pilots are more constrained for aerodynamic as well as regulatory reasons, they can often elect to climb above the rain where $N = S' = S = 0$.

IV. GENERALIZATION TO ARBITRARY TRAVEL PATHS AND MOVING RAINFALLS

The results obtained above can be extended to include the additional effects of (a) vertical motion of the surface element, and/or (b) rain falling askew the gravity vector (i.e., slanted or possessing a horizontal velocity component). Figure 3 illustrates the generalized geometry. The effect produced by traveling at any chosen vertical velocity $v_{z\ast}$, relative to $v_Z$ (which, one recalls, is directed opposite the vertical), is to change the horizontal surface volume element by the dimensionless ratio

$$f_{z\ast} = \left|\frac{v_{z\ast} - v_Z}{v_Z}\right| . \quad (8)$$
Similarly, a horizontally directed wind of magnitude $v_w$, making a constant angle $\psi$ relative to the direction $v_H$, will scale the vertical element by the ratio

$$f_W = \left| \frac{v_H - v_W \cos \psi}{v_H} \right|, \tag{9}$$

where the index $i$ denotes a given finite path element. The total number of strikes encountered over a monodirectional path element of length $D_i = (D_{H i} + D_{Z i})^{1/2}$ is given by

$$S_i(v_H, v_Z, v_W, v_{Z*}, \psi, \theta, D_i) = NA \left( f_Z + f_W \cos \theta \right) D_i \tag{10}$$

or

$$S_i(v_H, v_Z, v_W, v_{Z*}, \psi, \theta, D_i) = NA \left( 1 - \frac{v_{Z*}}{v_Z} \frac{\sin \theta}{v_H} \right) D_i + \left( 1 - \frac{v_W \cos \psi}{v_H} \right) \left( 1 - \frac{v_{Z*}}{v_Z} \frac{\cos \theta}{v_{Z*}} \right) D_i. \tag{11}$$

In practice, care must be taken here since wind directions are reported as the direction from which the wind originates, while horizontal velocity is directed in the direction of travel.

Notice that for $v_Z = 0$ or $v_W = 0$ the respective scaling factors go to unity and Eq. (11) collapses to Eq. (7). Further, if $v_Z = v_{Z*}$ or $v_W \cos \psi = v_H$ then the respective $f$ coefficients will vanish because such “free fall” conditions will null velocity against the rain field; the surface element will not strike any rain.

Thus to minimize the total number of strikes under these relaxed conditions, one would prefer $v_{Z*} = v_Z$, $v_W \cos \psi = v_H$. For aircraft, the former condition is temporarily satisfying (although in instrument conditions altitudes are assigned, requiring $v_{Z*} = 0$). Typically, pedestrians, automobiles, and ships are also constrained to surfaces where $v_{Z*} = 0$. In each of these cases, one can only control $v_H$, since $v_W \cos \psi$ is fixed for a monodirectional route. For convenience, let $v_{W*}$ denote $v_W \cos \psi$.

Under these conditions, there are two cases to discuss. If $v_{Z*} = 0$ and $\psi < \pi/2$, then $v_{W*} = 0$. Therefore, $v_H$ will be non-negative and $f_Z = 1$, $f_W < 1$. Under these circumstances, Eq. (11) reduces to

$$S_i = \left( \frac{\sin \theta}{v_H} + f_W \frac{\cos \theta}{v_Z} \right) NAD_i. \tag{12}$$

Hence, the minimization strategy is exactly as it was for the restricted problem discussed previously (since the net relative motion continues to oppose the wind), $f_Z > 1$ will make large $\theta$ (tilt) angles increasingly attractive. In the limit as $v_H \to \infty$, the number of rainstrikes will be driven to $S = NAD \cos \theta / v_Z$ over the path segment $i$.

On the other hand, if $\psi < \pi/2$, then $v_{W*} = 0$, $|f_W| < 1$, and the wind must be in the direction of the velocity vector. Thus Eq. (11) becomes

$$S_i = \left[ \frac{\sin \theta}{v_H} + f_W (\cos \theta / v_Z) \right] NAD_i. \tag{12a}$$

Should $\theta$ be fixed, the calculus gives the minimum number of strikes as $NAD / v_Z$; this solution is obtained when $\tan \theta = |v_W / v_Z|$. If one is free to orient $\theta$, traveling at $v_{W*} = v_H$ with $\theta = 0$ yields zero strikes. Another zero strike rate is obtained for the physically unrealizable set of conditions $\theta = \pi/2$, $v_H = \infty$. These results can be applied to minimize the rainstrikes over any curve or multilinear path in a constant isotropic rain.

Finally, throughout this paper we have enjoyed the assumption of the invariance of the intensity $N$ over time. This restriction can also be relaxed. The strategy suggested by Eq. (7) will remain valid should $N$ change to some new value $N'$ at some later time if the change is stochastic in time.

V. APPROXIMATING COMPLEX GEOMETRIES WITH FINITE ELEMENTS

To extend the results obtained in this paper to surfaces of arbitrarily complex geometry, an ensemble of $n$ surface elements (whose area need not be equal) may be summed to represent the required surface. Of course, as the surface element dimensions decrease, the actual surface geometry is represented with increasing fidelity. Since this requires a concomitant increase in $n$ (which should grow approximately as the inverse square of $A$), computing facilities will be required for calculations beyond the crudest surface element representations.

The functions $S'$ and $S$ will thus be given by

$$S'(v_H, v_{Z*}, v_{Z*}, v_W, \theta, \psi, A_i) = N \sum_{i=1}^{n} \left( \left| \frac{v_{Z*}}{v_Z} - 1 \right| \sin \theta_i + 1 - \frac{v_W \cos \psi}{v_H} \right) \frac{\cos \theta}{v_Z} A_i \tag{13}$$

817 Am. J. Phys., Vol. 51, No. 9, September 1983

S. A. Stern 817
and

\[ S(v_H, v_z, v_Z, v_W, \theta, \psi_i, D_i, A_i) \]

\[ = N \sum_{i=1}^{n} \left( \left| \frac{v_{Z,i}}{v_Z} \right| - 1 \right) \sin \theta_i \]

\[ + \left| 1 - \frac{v_W \cos \psi_i}{v_H} \right| \left| \frac{v_{H,i} \cos \theta_i}{v_Z} \right| A_i D_i. \]  

(13a)

For rigid bodies, \( \psi_i = \text{constant} \) for each element.

As an alternative to finite elements, one can often rather “easily” integrate around many surfaces which enjoy planes of symmetry. This technique is particularly suitable for rigid bodies (e.g., vehicles) where \( v_{Z,i} = v_Z \), and \( v_{H,i} = v_H \). Under this assumption, one can evaluate \( S^* \) and \( S \) for a body of arbitrarily complex geometry exactly as in Eqs. (12) and (12a) by simply employing the projected horizontal and vertical cross sections.

VI. CONCLUDING REMARKS

The problem of an optimal traverse speed through the rain is suitable for high school senior and undergraduate physics and mathematics students for several reasons. Chief among these is the analytic exercise they will receive in a problem they may have already casually considered. Although no calculus or vector mechanics is required to derive the solution and its generalizations, a thorough knowledge of algebra and analytical geometry is essential. Without rederiving Eq. (7) itself, there are several “stand-alone” facets of the problem; for example, a class could be assigned the task of generalizing the solution for a nonstationary rain field. Numerical approximations are also interesting (especially for computer programming classes). One can model the human body as a rectangular box and get reasonable results for a variety of human dimensions, rainfall terminal velocities, and \( \psi \) angles in a short time with only a personal calculator. One challenging exercise would be to solve the problem with spatially or temporarily relaxed constraints on \( N \).

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A criterion for stationary states in quantum mechanics

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We show a criterion for a vector \( \phi \) in a complex inner product space \( V \) to be an eigenvector of a Hermitian operator \( A \) in \( V \) in terms of the values of inner products \( \langle X \phi, A \phi \rangle \) for Hermitian operators \( X \). This result can be applied to the characterization of stationary states in quantum mechanics.

The state at time \( t \) of a conservative quantum-mechanical system \( S \) is determined by a state vector \( \psi \), which is an element of a complex Hilbert space \( H \). As a function of the parameter \( t \) the vector \( \psi \) is differentiable, and the time evolution of \( S \) is determined by the Schrödinger equation

\[ i \frac{\partial}{\partial t} \psi = A \psi, \quad (1) \]

where \( A \), assumed independent of \( t \), is a self-adjoint operator corresponding to the total energy of \( S \).

It is one of the postulates of quantum theory that the expected value of the results of measurements at time \( t \) of an observable of \( S \) may be expressed in the form

\[ f_X(t) = \langle \psi, X \psi \rangle, \quad (2) \]

where \( \langle \theta, \phi \rangle \) denotes the inner product (the convention adopted is the usual one of quantum mechanics, in which \( \langle \theta, \phi \rangle \) is linear in \( \theta \) and antilinear in \( \phi \)) of elements \( \theta \) and \( \phi \) in \( H \), and \( X \), again assumed independent of \( t \), is a Hermitian operator representing the observable.

From (1) and (2) and \( X \) being Hermitian it follows that

\[ i \frac{d}{dt} f_X(t) = \langle X \psi, A \psi \rangle - \langle A \psi, X \psi \rangle \]

\[ = 2i \Im \langle X \psi, A \psi \rangle. \]

818 Am. J. Phys. 51 (9), September 1983 © 1983 American Association of Physics Teachers 818